Changing Acceleration with Basic Dependencies

1 Dimensional Example:

$\sum F = f(t)$

$\sum F = ma$

$a = \frac{f(t)}{m}$

$\sum F = kt^2$

$a = \frac{k}{m} t^2$

$v = v_0, x = x_0$ at $t = 0$

$v(t) = ?$

$x(t) = ?$

$\frac{dv}{dt} = a = \frac{k}{m} t^2$

$\int dv = \frac{k}{m} \int t^2 \, dt$

$v = k/m \left(\frac{t^3}{3} + C \right)$

$v_0 = C$

$\boxed{v = k/3m \, t^3 + v_0}$

$\frac{dx}{dt} = \frac{k}{3m} t^2 + v_0$

$dx = \left(\frac{k}{3m} t^3 + v_0 \right) dt$

$\int dx = k/3m \int t^3 dt + v_0 \int dt$

$x = k/3m \, \frac{t^4}{4} + v_0 t + C$

$x_0 = C$

$\boxed{x(t) = \frac{k}{12m} t^4 + v_0 t + x_0}$

2 Dimensional Example:

At $t = 0$:

$v_0 = v_{0x} \hat{i} + v_{0y} \hat{j}$

$\sum F = -mg \hat{j}$

$a = -g \hat{j}$

$\vec{x} = x_0 \hat{i} + y_0 \hat{j}$

x: $a_x = 0$

$\frac{dv_x}{dt} = 0 \Rightarrow \int dv_x = 0 \Rightarrow v_x = C - 0$

$t = 0 \Rightarrow v_x = v_{0x}$

$\frac{dx}{dt} = v_{0x}$

$\int dx = v_{0x} \int dt$

$x = v_{0x} t + C \Rightarrow t = 0 : x_0 = C$

$\boxed{\begin{array}{l} x(t) = x_0 + v_{0x} t \\ v_x(t) = v_{0x} \end{array}}$

$v(t) = v_{0x} \hat{i} + (v_{0y} - gt) \hat{j}$

$\vec{x}(t) = x(t) \hat{i} + y(t) \hat{j}$

$\vec{x}(t) = (x_0 + v_{0x} t) \hat{i} + \; y_0 + v_{0y} t - \frac{1}{2} g t^2$

y:

$a_y = -g$

$\frac{dv_y}{dt} = -g$

$\int dv_y = -g \int dt$

$v_y = -gt + C$

$t = 0, v_y = v_{0y} \Rightarrow v_{0y} = C$

$v_y(t) = v_{0y} - gt$

$\frac{dy}{dx} = v_{0y} - gt$

$\int dy = v_{0y} \int dt - g \int t \, dt$

$y = v_{0y} t - g \frac{t^2}{2} + C \Rightarrow t = 0, C = y_0$

$\boxed{\begin{array}{l} y(t) = y_0 + v_{0y} t - \frac{1}{2} g t^2 \\ v_y(t) = v_{0y} - gt \end{array}}$

Ex.

$$\sum F = -4t^2\,\hat{i} + 3e^{-4t}\,\hat{j}$$

$$t=0,\ \vec{x}_0 = x_0\hat{i} + y_0\hat{j}$$

$$v_0 = v_{0x}\hat{i} + v_{0y}\hat{j}$$

$$a(t) = \frac{-4}{m}t^2\,\hat{i} + \frac{3e^{-4t}}{m}\,\hat{j}$$

$$a_x(t) = \frac{-4}{m}t^2$$

$$\frac{dv_x}{dt} = \frac{-4}{m}t^2$$

$$\int dv_x = -4/m\int t^2\,dt$$

$$v_x = -4/m\,\frac{t^3}{3} + C$$

$$v_{0x} = C$$

$$v_x(t) = v_{0x} - \frac{4}{3m}t^3$$

$$\frac{dx}{dt} = v_{0x} - \frac{4}{3m}t^3$$

$$\int dx = v_{0x}\int dt - 4/3m\int t^3\,dt$$

$$x = v_{0x}t - 4/3m\left(\frac{t^4}{4}\right) + C$$

$$x = v_{0x}t - \frac{t^4}{3m} + C$$

$$C = x_0$$

$$x(t) = v_{0x}t - \frac{t^4}{3m} + x_0$$

$$a_y(t) = \frac{3e^{-4t}}{m}$$

$$\frac{dv_y}{dt} = \frac{3e^{-4t}}{m}$$

$$\int dv_y = \frac{3}{m}\int e^{-4t}\,dt$$

$$v_y = \frac{3}{m}\left(\frac{e^{-4t}}{-4}\right) + C$$

$$t=0 \Rightarrow v_{0y} = \frac{-3}{4m} + C$$

$$C = v_{0y} + 3/4m$$

$$v_y = \frac{-3e^{-4t}}{4m} + v_{0y} + 3/4m$$

$$v_y(t) = v_{0y} + 3/4m\left(1 - e^{-4t}\right)$$

$$\frac{dy}{dt} = v_{0y} + 3/4m\left(1 - e^{-4t}\right)$$

$$\int dy = v_{0y}\int dt + \frac{3}{4m}\int dt - 3/4m\int e^{-4t}\,dt$$

$$y = \left(v_{0y} + \frac{3}{4m}\right)t + \frac{3e^{-4t}}{16m} + C$$

$$t=0 \Rightarrow y_0 = \frac{3}{16m} + C \Rightarrow C = y_0 - \frac{3}{16m}$$

$$y(t) = y_0 - \frac{3}{16m} + \left(v_{0y} + \frac{3}{4m}\right)t + \frac{3e^{-4t}}{16m}$$

Derivation of integral of ln(x)

u, v

$$\int u\,dv = u \cdot v - \int v\,du$$

$$d f(x) = \frac{d f(x)}{dx}\,dx$$

$$\int \ln x\,dx = x \ln x \int x\,d(\ln x)$$

$u = \ln x$
$v = x$

$$= x \ln x - \int x \left(\frac{1}{x}\right) dx$$

$$= x \ln x - x$$

Made with Goodnotes

Accelerations Caused by Force that depends on velocity

Drag Force:

- Typically an example of a force that depends on velocity

$$a = A - Bv^2$$

$$t = 0 \rightarrow v = 0,\ x = x_0$$

$$\frac{dv}{dt} = A - Bv^2$$

$$\int \frac{dv}{(A - Bv^2)} = \int dt$$

$$\int \frac{dv}{(A/B - v^2)} = B \int dt$$

$$A/B = k^2$$

$$\int \frac{dv}{(k^2 - v^2)} = B \int dt$$

$$\int \frac{dv}{(k-v)(k+v)} = B \int dt$$

$$\frac{C}{k-v} + \frac{D}{k+v} = \frac{1}{(k-v)(k+v)}$$

$$\frac{C(k+v) + D(k-v)}{(k+v)(k-v)} = \frac{1}{(k-v)(k+v)}$$

$$k(C + D) + v(C - D) = 1$$

$$C - D = 0$$

$$C = D$$

*b/c no v term on right

$$k(C + D) = 1$$

$$k(2D) = 1$$

$$D = \frac{1}{2k} = C$$

$$\frac{1}{(k-v)(k+v)} = \frac{1}{2k}\left(\frac{1}{k-v} + \frac{1}{k+v}\right)$$

$$\frac{1}{2k}\left(\int \frac{dv}{k-v} + \int \frac{dv}{k+v}\right) = B \int dt$$

$$\frac{1}{2k}\left(-\ln(k-v) + \ln(k+v)\right) = Bt + C$$

At $t = 0$

$$1/2k(-\ln k + \ln k) = +C \Rightarrow C = 0$$

$$1/2k\left(\ln(k+v) - \ln(k-v)\right) = Bt$$

$$\ln(k+v) - \ln(k-v) = 2kBt$$

$$e^{\ln\left(\frac{k+v}{k-v}\right)} = e^{2kBt}$$

$$\frac{k+v}{k-v} = e^{2kBt}$$

$$k+v = (k-v)e^{2kBt}$$

$$k+v = ke^{2kBt} - ve^{2kBt}$$

$$v(1 + e^{2kBt}) = k(e^{2kBt} - 1)$$

$$v = \frac{-k(1 - e^{2kBt})}{(1 + e^{2kBt})}$$

$$v = -\sqrt{\frac{A}{B}}\left(\frac{1 - e^{2\sqrt{\frac{A}{B}}Bt}}{1 + e^{2\sqrt{\frac{A}{B}}Bt}}\right)$$

$$\boxed{v = -\sqrt{\frac{A}{B}}\left(\frac{1 - e^{2\sqrt{AB}\,t}}{1 + e^{2\sqrt{AB}\,t}}\right)}$$

Forces

Newtons First Law:

- Object at rests stays at rest and object in uniform motion stays that way unless acted on by unbalanced force
- Forces can cause change in state of motion
- Idea of inertia: objects resist change

Newtons Second Law:

- Force is equal to rate of change of momentum

$$\sum F = \frac{d\vec{p}}{dt}$$

$$m = \text{const}$$

$$\sum F = \frac{d}{dt}(mv)$$

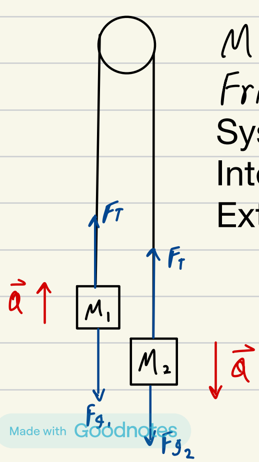
$$\sum F = ma$$

Newtons Third Law;

- Every action force has equal and opposite reaction force

System:

- System of objects: objects have to be connected in some way and they all have to have SAME acceleration



$M_2 > M_1$

Frictionless pully

System: M1, M2 and String

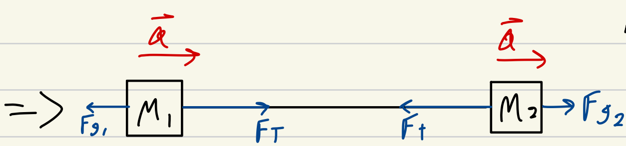
Internal Forces: Tension

External Forces: Gravity

$$\sum F_{external} = F_{g_2} - F_{g_1}$$

$$M_2 g - M_1 g = (M_1 + M_2) a$$

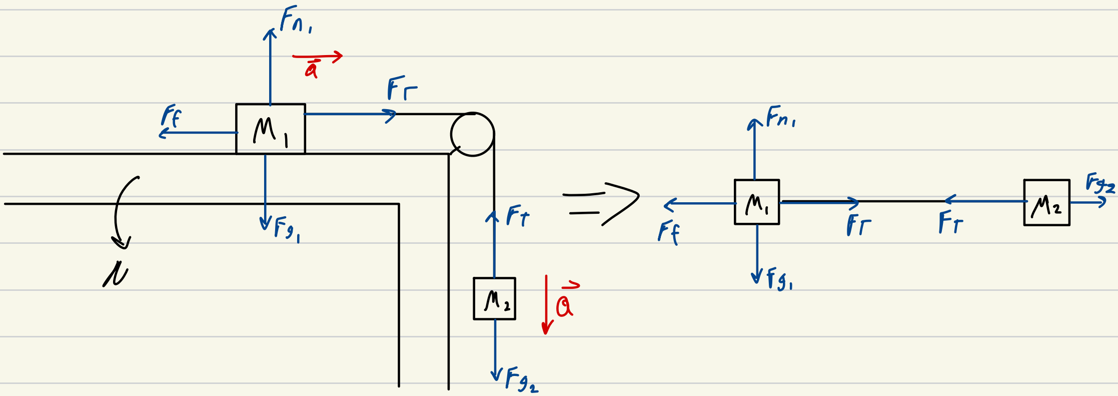
$$\boxed{a = \frac{(M_2 - M_1) g}{M_1 + M_2}}$$



$$F_T - M_1 g = M_1 a$$

$$\boxed{F_T = M_1(a + g)}$$

Ex.



$\sum F_\perp = 0$

$F_{n_1} = F_{g_1}$

$F_{n_1} = M_1 g$

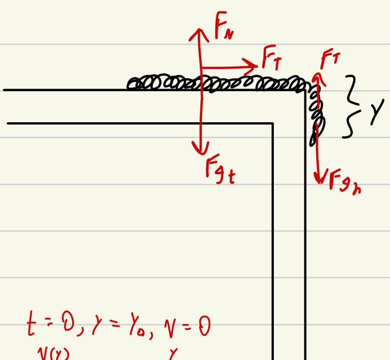
$F_f = \mu(M_1 g)$

$\sum F_\parallel = F_{g_2} - F_f$

$(M_1 + M_2) a = M_2 g - \mu(M_1 g)$

$$\boxed{a = \frac{M_2 g - \mu(M_1 g)}{(M_1 + M_2)}}$$

Ex. Cable of length L and uniform mass M



$\sum F_\parallel = ma$

$F_{g_h} = m_{sys} a$

Linear mass density: mass/length $= \lambda$

$\frac{M}{L} y g = Ma$

$a = \frac{g}{L} y$

$\frac{dv}{dt} = \frac{g}{L} y$

$\frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{g}{L} y$

$\frac{dv}{dy} \cdot v = \frac{g}{L} y$

$\int v \, dv = \int \frac{g}{L} y \, dy$

$\frac{v^2}{2} = \frac{g y^2}{2L} + C$

$$\boxed{v = \sqrt{\frac{g y^2}{L}}}$$

$+C = 0$ at $t = 0, v = 0, y = 0$

Alternate ←

$t = 0, y = y_0, v = 0$

$\int_0^{v(y)} v \, dv = \frac{g}{L} \int_{y_0}^{y} y \, dy$

$\left[\frac{v^2}{2}\right]_0^{v(y)} = \left[\frac{g}{L} \frac{y^2}{2}\right]_{y_0}^{y}$

$\frac{(v(y))^2}{2} - 0 = \frac{g}{L}\left(\frac{y^2}{2} - \frac{y_0^2}{2}\right)$

$(v(y))^2 = \frac{g}{L}(y^2 - y_0^2)$

$v(y) = \sqrt{\frac{g}{L}(y^2 - y_0^2)}$

$\frac{dy}{dt} = \sqrt{\frac{g}{L}} y$

$dy = \sqrt{g/L} \, y \, dt$

$\int \frac{1}{y} dy = \int \sqrt{g/L} \, dt$

$\ln y = \sqrt{g/L} \, t + C$

$t = 0, y = \varepsilon$ (Where $\varepsilon \to 0$)

$\ln y - \ln \varepsilon = \sqrt{g/L} \, t$

$\ln(y/\varepsilon) = \sqrt{g/L} \, t$

$y/\varepsilon = e^{\sqrt{g/L} \, t}$

$$\boxed{y = \varepsilon e^{\sqrt{g/L} \, t}}$$

Finding y as a function of time

$$\frac{dy}{dt} = \sqrt{g/L\,(y^2 - y_0)}$$

$$\int \frac{dy}{\sqrt{y^2 - y_0^2}} = \int \sqrt{\frac{g}{L}}\, dt$$

$$y = y_0 \sec\theta$$

$$y^2 - y_0^2 = y_0^2 \sec^2\theta - y_0^2$$

$$= y_0^2(\sec^2\theta - 1)$$

$$= y_0^2(\tan^2\theta)$$

$$\frac{dy}{dt} = y_0 \sec\theta \tan\theta\, d\theta$$

$$\int\left(\frac{y_0 \sec\theta \tan\theta\, d\theta}{\sqrt{y_0^2 \tan^2\theta}}\right) = \sqrt{g/L}\, dt$$

$$\int \sec\theta\, d\theta = \sqrt{g/L}\,t + C$$

$$\int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\, d\theta = \sqrt{g/L}\, t + C$$

$$u = \sec\theta + \tan\theta$$

$$du = (\sec\theta \tan\theta + \sec^2\theta)\, d\theta$$

$$\int \frac{du}{u} = \sqrt{\frac{g}{L}}\, t + C$$

$$\ln u = \sqrt{g/L}\, t + C$$

$$\ln(\sec\theta + \tan\theta) = \sqrt{g/L}\, t + C$$

$$\ln\left(\frac{y}{y_0} + \sqrt{\left(\frac{y}{y_0}\right)^2 - 1}\right) = \sqrt{g/L}\, t + C$$

$$t = 0,\ y = y_0$$

$$\left(\frac{y}{y_0} + \sqrt{\frac{y^2}{y_0^2} - 1}\right) = e^{\sqrt{g/L}\, t}$$

$$\frac{y}{y_0} + \frac{1}{y_0}\sqrt{y^2 - y_0^2} = e^{\sqrt{g/L}\, t}$$

$$\frac{y + \sqrt{y^2 - y_0^2}}{y_0} = e^{\sqrt{g/L}\, t}$$

$$y + \sqrt{y^2 - y_0^2} = y_0 e^{\sqrt{g/L}\, t}$$

$$\sqrt{y^2 - y_0^2} = y_0 e^{\sqrt{g/L}\, t} - y$$

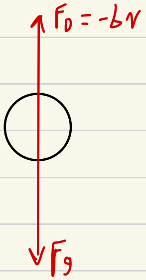
$$y^2 - y_0^2 = y_0^2 e^{2\sqrt{g/L}\, t} - 2yy_0 e^{\sqrt{g/L}\, t} + y^2$$

$$2yy_0 e^{\sqrt{g/L}\, t} = y_0^2\left(1 + e^{2\sqrt{g/L}\, t}\right)$$

$$y = \frac{1}{2} y_0 e^{-\sqrt{g/L}\, t}\left(1 + e^{2\sqrt{g/L}\, t}\right)$$

$$\boxed{y = \frac{y_0}{2}\left(e^{-\sqrt{g/L}\, t} + e^{\sqrt{g/L}\, t}\right)}$$

Simple Drag Force:



$$t = 0, \quad v = v_0$$

$$\Sigma F = F_g - F_D$$

$$mg - bv = ma$$

$$\frac{mg - bv}{m} = \frac{dv}{dt}$$

$$dt = \frac{m\,dv}{mg - bv} \qquad u = mg - bv$$

$$du = -b\,dv$$

$$\int dt = \int \frac{m\,du}{-bu} \qquad dv = du/-b$$

$$t = \frac{-m}{b} \ln u + C$$

$$t = \frac{-m}{b} \ln(mg - bv) + C$$

At $t = 0$

$$C = m/b \ln(mg - bv_0)$$

$$t = -m/b \ln(mg - bv) + m/b \ln(mg - bv_0)$$

$$t = -m/b \left(\ln(mg - bv) - \ln(mg - bv_0)\right)$$

$$-b/m\, t = \ln\left(\frac{mg - bv}{mg - bv_0}\right)$$

$$e^{-b/m\, t} = \frac{mg - bv}{mg - bv_0}$$

$$(mg - bv_0)e^{-\frac{bt}{m}} - mg = -bv$$

$$\frac{-(mg - bv_0)e^{-\frac{bt}{m}} + mg}{b} = v$$

$$v(t) = \frac{mg}{b} - \frac{(mg\,e^{-\frac{bt}{m}} + bv_0 e^{-\frac{bt}{m}})}{b}$$

$$\boxed{v(t) = \frac{mg}{b} - \frac{me^{-\frac{bt}{m}}}{b}\left(g - \frac{bv_0}{m}\right)}$$

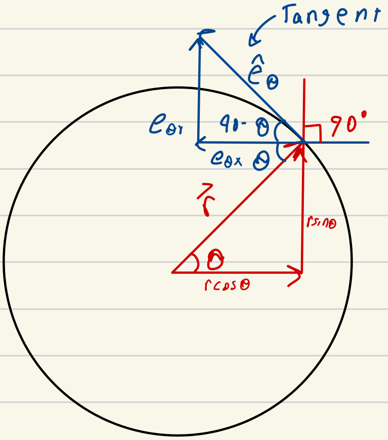
$$\frac{mg}{b} = V_T = \text{terminal velocity}$$

Newtons Laws Applied to Circular Motion

$\dot{a}$, $\ddot{a}$ 1st/2nd derivative with respect to time

a', a'' 1st/2nd derivative with respect to special coordinates

Derivation for Centripetal Acceleration



$\vec{r} = x\hat{i} + y\hat{j}$

$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j}$

$\hat{e}_r = \frac{\vec{r}}{r} = \cos\theta\hat{i} + \sin\theta\hat{j}$

↑

Unit Vector

$\hat{e}_{\theta x} = |\hat{e}_\theta|\cos(90-\theta)(-\hat{i})$

$\hat{e}_{\theta y} = |\hat{e}_\theta|\sin(90-\theta)(\hat{j})$

$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$

$\hat{e}_r \cdot \hat{e}_\theta$ dot prod. should be 0 b/c orthogonal

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$\vec{r} = r\hat{e}_r$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr\hat{e}_r}{dt}$

$\vec{v} = \frac{\hat{e}_r\,dr}{dt} + \frac{r\,d\hat{e}_r}{dt}$

$= \hat{e}_r\dot{r} + r\frac{d}{dt}(\cos\theta\hat{i} + \sin\theta\hat{j})$

$= \hat{e}_r\dot{r} + r(-\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j})$

$= \hat{e}_r\dot{r} + r\dot{\theta}(-\sin\theta\hat{i} + \cos\theta\hat{j})$

$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\frac{d\vec{v}}{dt} = a$

$\vec{a} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$

$\vec{a} = \frac{d\dot{r}}{dt}\hat{e}_r + \dot{r}\frac{d}{dt}\hat{e}_r + \frac{dr}{dt}\dot{\theta}\hat{e}_\theta + r\frac{d\dot{\theta}}{dt}\hat{e}_\theta + r\dot{\theta}\frac{d\hat{e}_\theta}{dt}$

$= \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}(\cos\theta\hat{i} + \sin\theta\hat{j}) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}(-\sin\theta\hat{i} + \cos\theta\hat{j})$

$= \ddot{r}\hat{e}_r + \dot{r}(-\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j}) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j})$

$= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}(-\sin\theta\hat{i} + \cos\theta\hat{j}) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}^2(-\cos\theta\hat{i} - \sin\theta\hat{j})$

$= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$

Circular Motion

$r \to \text{const} \therefore \dot{r} = \ddot{r} = 0$

$r\ddot{\theta} \to$ Tangential Accel

$2\dot{r}\dot{\theta} \to$ Corioli's Acceleration

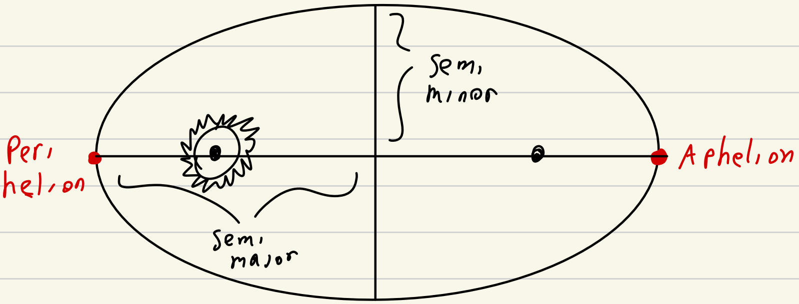
$-r\dot{\theta}^2\hat{e}_r \to$ Centripetal acceleration

$\vec{a} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta$

↓

$\omega^2 r\hat{e}_r \to \frac{v^2}{r^2}r\hat{e}_r \to \frac{v^2}{r}$

Elliptical Orbit Terminiology:

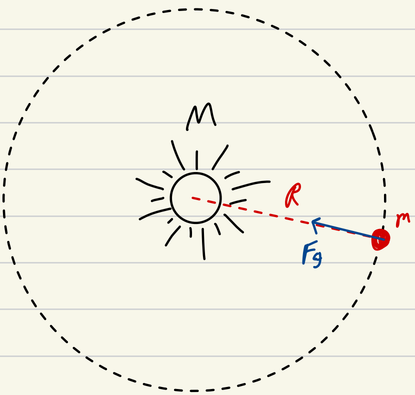


→ Planet

→ Sun

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

Circular Orbit:



$$\Sigma F_{rad} = F_g$$

$$\frac{mv^2}{R} = \frac{GmM}{R^2}$$

$$v^2 = \frac{gM}{R}$$

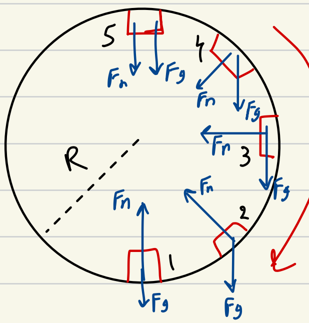
$$\frac{4\pi^2 r^2}{T^2} = \frac{GM}{R}$$

$$\frac{4\pi^2 R^3}{GM} = T^2$$

$$T^2 \propto R^3$$

Vertical Circle:

Roller coaster



1) $F_n - F_g = \frac{mv^2}{R}$

$F_n = m\left(\frac{v^2}{R} + g\right)$

2) $F_n - F_g \cos\theta = m v^2/R$

$F_n = m\left(\frac{v^2}{R} + g\cos\theta\right)$

3) $F_n = \frac{mv^2}{R}$

4) $F_n + F_g \sin\theta = m v^2/R$

$F_n = m\left(v^2/R - g\sin\theta\right)$

5) $F_n + F_g = \frac{mv^2}{R}$

$F_n = m\left(\frac{v^2}{R} - g\right)$

Center of mass formula (discrete):

$$r_{cm} = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i}$$

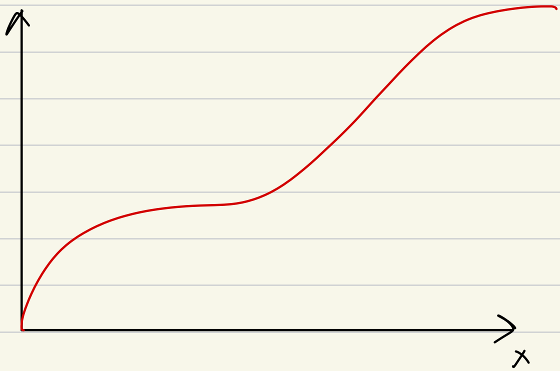
Work

Work:

- Force dot product displacement
- Constant Force:

$$W = \vec{F} \cdot \vec{\Delta x} = F \Delta x \cos\theta$$

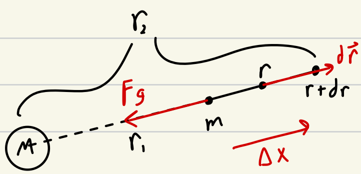
- Work measured in Joules
- For changing force:



$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$
$$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$W = \int_{x_i}^{x_f} \vec{F}\, dr = \int_{x_1}^{x_f} F_x\, dx + \int_{x_1}^{x_f} F_x\, dx + \int_{x_1}^{x_f} F_x\, dx$$

Ex.



$$F_g(r) = \frac{GmM}{r^2}\hat{r}$$

$$dW = \vec{F_g(r)} \cdot d\vec{r}$$
$$dW = -\frac{GmM}{r^2}\hat{r} \cdot dr\,\hat{r}$$
$$= -\frac{GmM}{r^2}\, dr\,(\hat{r} \cdot \hat{r})$$
$$\int dW_g = \int -\frac{GmM}{r^2}\, dr$$
$$W_g = -GmM \int \frac{dr}{r^2}$$
$$W_g = -GmM\left(-\frac{1}{r}\right)\Big|_{r_1}^{r_2}$$
$$W_g = GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Potential Energy

- Work done by conservative force is path independent
- Potential energy emerged from conservative forces

$$\Delta U_g = -W_g$$

$$\Delta U_g = \frac{GmM}{r_1} - \frac{GmM}{r_2} = U_{gf} - U_{gi}$$

$$U_g = \frac{-GmM}{r}$$

$$V_g = \frac{-GM}{r}$$

$$U_g = m V_g$$

$$dU = -\vec{F} \cdot d\vec{r}$$

$$\vec{F} = -\vec{\nabla} U$$

$$\vec{\nabla} U = \hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}$$

↳ Gradient operator

Energy

$\sum \vec{F},\ \Delta \vec{r}$

$$\therefore \sum W = \sum \vec{F} \cdot \Delta \vec{r} = m\vec{a} \cdot \Delta \vec{r} = \Delta KE$$

$\downarrow$

$$\sum W_c + \sum W_{nc} = E_{kf} - E_{ki}$$

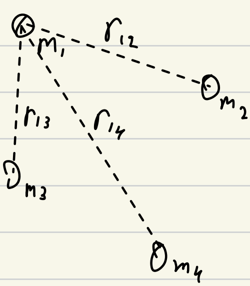
$$\sum -(\Delta U_c) + \sum W_{nc} = E_{kf} - E_{ki}$$

$$\sum (U_{ci} - U_{cf}) + \sum W_{nc} = E_{kf} - E_{ki}$$

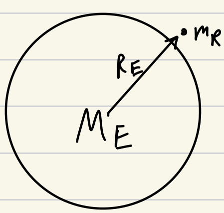
$$\sum U_{ci} + E_{ki} + \sum W_{nc} = E_{kf} + \sum U_{cf}$$

$$ME_i + \sum W_{nc} = ME_f$$

Ex.



$$U_{g\,sys} = -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_1m^2}{r_{14}} + \ldots r_{34}$$



$r_i = R_E \qquad r_f \to \infty$

$v_i = v_E \qquad v_f = 0\ m/s$

$$U_{gi} = -\frac{GM_E m_R}{R_E} \qquad U_{gf} = 0$$

$$E_{ki} = \tfrac{1}{2} m_R v_E^2 \qquad E_{kf} = 0$$

$$\sum W_{nc} = 0$$

$$\tfrac{1}{2}\cancel{m_R} v_E^2 + \frac{-G\cancel{m_R} m_E}{r_E} = 0$$

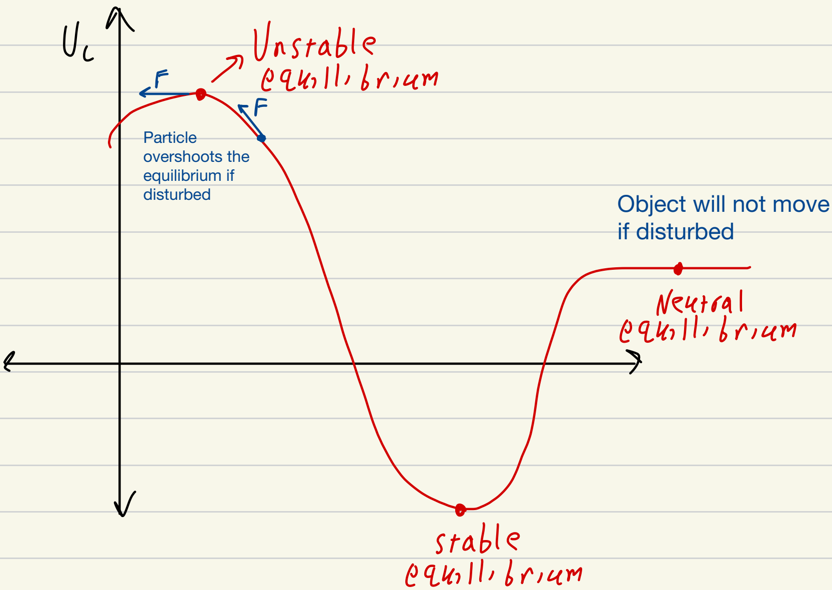
$$\boxed{v_E^2 = \frac{2GM}{R_E} \quad \text{this velocity would escape Earth}}$$

Quick Recap:

$$W = \int \vec{F} \cdot d\vec{r}$$
$$W_c = -\Delta U_c$$
$$\vec{F}_c = -\vec{\nabla} U_c$$

$$U_g = \frac{-GmM}{r}$$
$$F_g = \frac{-dU_g}{dr}\hat{r}$$
$$= -\hat{r}GmM\frac{d}{dr}\left(\frac{1}{r}\right)$$
$$F_g = \frac{-\hat{r}GmM}{r^2}$$

$$F_c = -\frac{dU_c}{dr}$$

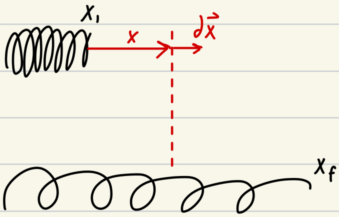


- All Maxima are unstable
- All Minima are stable
- All flat are neutral

Deriving Spring Potential Energy:

$F_s = -kx\hat{i}$

$d\vec{x} = d\vec{x}\,\hat{i}$



$dW = \vec{F_s} \cdot d\vec{x} = -kx\hat{i} \cdot d\vec{x}\hat{i} = -kx\,dx$

$\int dW = \int_{x_f}^{x_i} -kx\,dx$

$W = \left[\frac{-kx^2}{2}\right]_{x_f}^{x_1}$

$W_s = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$

$W_s = -\Delta U_s$

$W_s = U_1 - U_f$

$U_s = \frac{1}{2}kx^2$

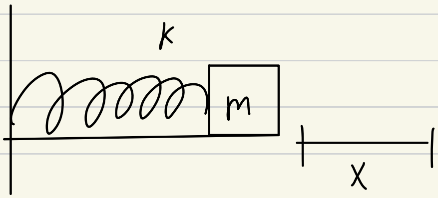
Power:

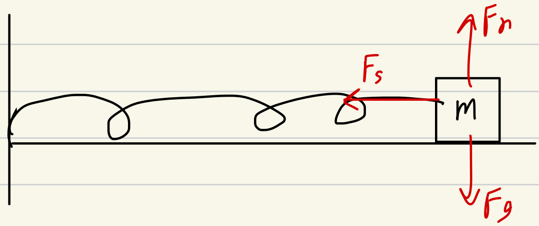
$P = \frac{dW}{dt} = \frac{dE}{dt}$

$P = \frac{\vec{F} \cdot d\vec{x}}{dt} = \vec{F} \cdot \vec{v}$

Simple Harmonic Motion

Consider a spring:





$$\sum F_{\perp} = 0$$
$$\therefore F_n = F_g$$

$$\sum F_{\parallel} = ma$$
$$F_s = ma$$
$$-kx = ma$$

$$a = \frac{-k}{m} x$$
$$\frac{dv}{dt} = \frac{-k}{m} x$$
$$\frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{-k}{m} x$$
$$v \frac{dv}{dx} = \frac{-k}{m} x$$
$$v\, dv = \frac{-k}{m} x\, dx$$
$$t = 0,\ x = x_0,\ v = 0$$
$$\int v\, dv = \int \frac{-k}{m} x\, dx$$
$$\frac{v^2}{2} = \frac{-kx^2}{2m} + C$$
$$C = \frac{kx_0^2}{2m}$$
$$\frac{1}{2} v^2 = \tfrac{1}{2} \tfrac{k}{m} (x_0^2 - x^2)$$
$$\tfrac{1}{2} m v^2 = \tfrac{1}{2} k x_0^2 - \tfrac{1}{2} k x^2$$
$$KE_f + U_{sf} = U_{si}$$

$$v = \sqrt{\frac{k}{m} (x_0^2 - x^2)}$$
$$\frac{dx}{dt} = \sqrt{\frac{k}{m} (x_0^2 - x^2)}$$
$$\int \frac{dx}{\sqrt{x_0^2 - x^2}} = \int \sqrt{k/m}\, dt$$
$$\frac{1}{x_0} \int \frac{dx}{\sqrt{1 - x^2/x_0^2}} = \sqrt{k/m} \int dt$$
$$u = x/x_0$$
$$du = dx/x_0$$
$$\int \frac{du}{\sqrt{1-u^2}} = \sqrt{k/m}\, t$$
$$\sin^{-1}(u) = \sqrt{k/m}\, t + C$$
$$\sin^{-1}(x/x_0) = \sqrt{k/m}\, t + C$$
$$t = 0,\ x = x_0$$
$$\sin^{-1}(1) = C$$
$$C = \pi/2$$
$$x/x_0 = \sin\left(\sqrt{k/m}\, t + \pi/2\right)$$
$$x(t) = x_0 \sin\left(\sqrt{k/m}\, t + \pi/2\right)$$
$$x(t) = x_0 \cos\left(\sqrt{k/m}\, t\right)$$

Simple Harmonic Oscillator general equation:

$$x(t) = A\cos\left(\sqrt{k/m}\,t\right) + B\sin\left(\sqrt{k/m}\,t\right)$$

Another Derivation:

$$ma = -kx$$
$$a = \frac{-k}{m}x$$
$$\frac{k}{m} = w^2$$
$$a = -w^2 x$$
$$\frac{d^2x}{dt^2} = -w^2 x$$
$$\frac{d^2x}{dt^2} + w^2 x = 0$$
$$\left(\frac{d^2}{dt^2} + w^2\right)x = 0$$
$$\left(\frac{d}{dt} + iw\right)\left(\frac{d}{dt} - iw\right)x = 0$$

$$\left(\frac{d}{dt} - iw\right)x = 0$$
$$\left(\frac{d}{dt} + iw\right)x = 0$$

$$\frac{dx}{dt} = -iwx \rightarrow \int\frac{dx}{x} = -iw\int dt \rightarrow \ln x = -iwt + C$$
$$x = Ae^{-iwt}$$

$$\frac{dx}{dt} = iwx \rightarrow \ln x = iwx + D$$
$$x = Be^{iwt}$$

$x(t) = Ae^{-iwt} + Be^{iwt} \rightarrow$ linear combination of solutions

$$V(t) = -iwAe^{-iwt} + iwBe^{iwt}$$
$$t = 0,\ x = x_0,\ v = 0$$

$$x_0 = A + B$$
$$0 = -iwA + iwB$$
$$0 = B - A$$

$$x_0 = 2B$$
$$B = \frac{x_0}{2} = A$$

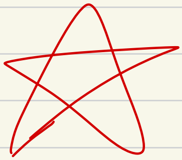
$$x(t) = x_0\left(\frac{e^{-iwt} + e^{iwt}}{2}\right)$$
$$= x_0\cos(wt)$$

Simple Harmonic Oscillator #2:

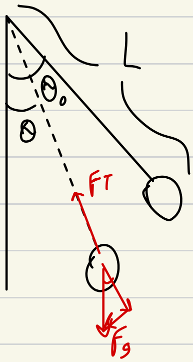
$$\frac{d^2x}{dt^2} = -\omega^2 x$$

$$T = \frac{2\pi}{\omega}$$

$$\omega = \sqrt{k/m}$$



Pendulum Ex.



$t = 0$

$\theta = \theta_0$

$v = 0$

$\Sigma F_{rad} = T - F_g \cos\theta$

$\Sigma F_t = -F_g \sin\theta$

$\theta = s/L$

$s = L\theta$

↑
Arc length

$v_t = \frac{ds}{dt}$

$$\cancel{m}\frac{dv_t}{dt} = -\cancel{m}g\sin\theta$$

$$\frac{d^2s}{dt^2} = -g\sin\theta$$

$$L\frac{d^2\theta}{dt^2} = -g\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Not simple harmonic oscillator b/c not linear

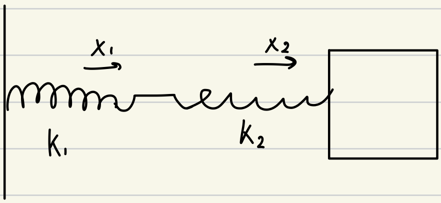
However, in small angle approximation:

$$\sin\theta = \theta$$

$$\frac{d^2\theta}{dt^2} = \frac{-g}{L}\theta$$

Springs in Parallel/Series

Series:



$$k_{effective} = k_e$$

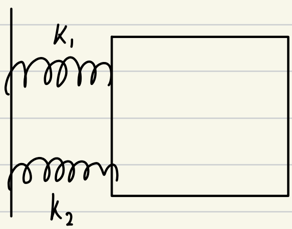
$$k_1 x_1 = k_2 x_2 = k_e (x_1 + x_2)$$

$$x_2 = \frac{k_1 x_1}{k_2}$$

$$k_1 x_1 = k_e \left(x_1 + \frac{k_1 x_1}{k_2}\right)$$

$$\boxed{\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2}}$$

Parallel:

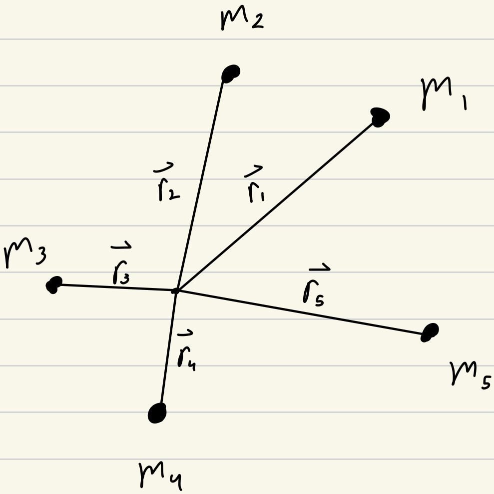


$$k_e x = k_1 x + k_2 x$$

$$\boxed{k_e = k_1 + k_2}$$

Linear Momentum

Center of Mass (Discrete set of masses):



$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

$$x_{cm} = \frac{\sum m_i x_i}{\sum m_i}$$

$$y_{cm} = \frac{\sum m_i y_i}{\sum m_i}$$

$$\sum_{i=1}^{n}$$

For a Rigidbody/Extended Object (ex. A cone):

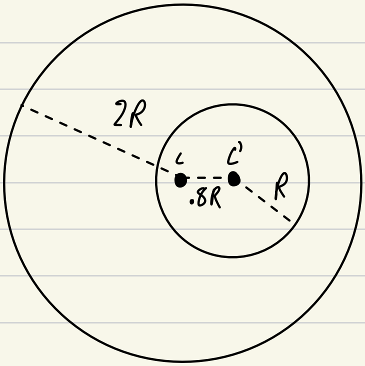
$$\vec{r}_{cm} = \frac{\int dm\, \vec{r}}{\int dm}$$

Mass can also be represented through density

$$m = \rho V$$

Made with Goodnotes

Advanced Examples)



$\sigma = \frac{M}{A}$

2-D Density

Large Disk

$$M_L = \sigma A$$
$$= \sigma(\pi(2R)^2)$$
$$= 4\pi R^2 \sigma$$

Small Disk)

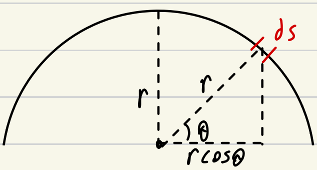
Treat empty area as negative mass

$$M_s = -\sigma A$$
$$= -\pi R^2 \sigma$$

Using C as origin

$$X_{cm} = \frac{M_L r_L + M_s r_s}{M_L + M_s}$$
$$= \frac{0 - \cancel{\sigma}\cancel{\pi}\cancel{R^2}(.8R)}{4\cancel{\pi}\cancel{R^2}\cancel{\sigma} - \cancel{\sigma}\cancel{\pi}R^2}$$
$$\boxed{= \frac{-.8R}{3}}$$

Semi-circular Wire)



$\lambda = \frac{M}{L}$

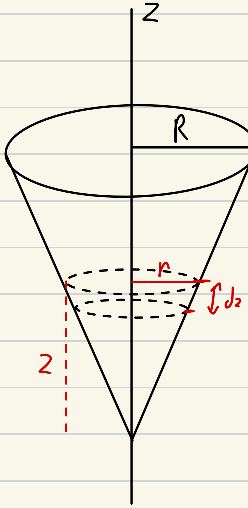
$$X_{cm} = \frac{\int \lambda \, d\theta \cdot r\cos\theta}{\int \lambda \, ds} = \frac{\lambda r^2 \int_0^\pi \cos\theta \, d\theta}{\lambda \int_0^\pi ds}$$

$$= \frac{\left[\lambda r^2 \sin\theta\right]_0^\pi}{\lambda r \pi} = 0$$

$$Y_{cm} = \frac{\int \lambda \, ds \, r\sin\theta}{\int \lambda \, ds} = \frac{\lambda r^2 \int_0^\pi \sin\theta \, d\theta}{\lambda r \pi} = \frac{r}{\pi}\left[-\cos\theta\right]_0^\pi = \frac{2r}{\pi}$$

Cone)

Divide into disks of thickness dz



$$V = \frac{1}{3}\pi R^2 h \qquad dm = \rho\, dV$$

$$\rho = M/\tfrac{1}{3}\pi R^2 h \qquad dm = \rho \pi r^2 dz$$

$$\frac{dm}{dV} = \rho \qquad r = \frac{z}{h} R$$

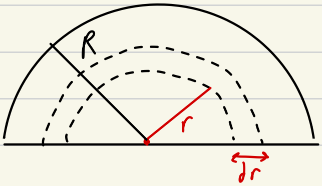
$$\frac{h}{R} = \frac{z}{r} \qquad dm = \rho\pi \frac{z^2}{h^2} R^2 dz$$

$$z_{cm} = \frac{\int dm\, z}{\int dm} = \frac{\int_0^h \cancel{\rho\pi} \frac{z^2}{\cancel{h^2}} \cancel{R^2} dz \cdot z^3}{\int_0^h \cancel{\rho\pi} \frac{z^2}{\cancel{h^2}} \cancel{R^2} dz}$$

$$= \frac{\int z^3 dz}{\int z^2 dz} = \frac{\frac{z^4}{4}}{\frac{z^3}{3}}$$

$$= \boxed{\frac{3h}{4}}$$

Semi-Circular Plate)



Cut up into tiny wires

$$x_{cm} = 0$$
$$y_{cm} = \frac{2r}{\pi}$$
$$\sigma = \frac{M}{A} = \frac{2M}{\pi R^2}$$

$$dm = \sigma\, dA$$
$$dm = \sigma \pi r\, dr$$

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$
$$dA = 2\pi r\, dr$$

$\Downarrow$ half

$$dA = \pi r\, dr$$

$$y_{cm} = \frac{\int dm\, y}{\int dm} = \frac{\int_0^R \sigma \pi r\, dr \cdot \frac{2r}{\pi}}{\int_0^R \sigma \pi r\, dr}$$

$$= \frac{2\sigma \int_0^R r^2 dr}{\sigma \frac{\pi R^2}{2}} = \frac{2\cancel{\sigma} \frac{R^3}{3}}{\frac{\cancel{\sigma} \pi R^2}{2}} = \boxed{\frac{4R}{3\pi}}$$

Impulse and Momentum:

$$\vec{F}_{sys} = \frac{d\vec{P}_{sys}}{dt} \qquad \vec{P}_{sys} = \sum_{i=1}^{n} m_i \vec{V}_i$$

Impulse Momentum Thm.

$$F_{sys}\, dt = d\vec{P}_{sys}$$

$$\int_0^t F_{sys}\, dt = \int_{P_1}^{P_f} d\vec{P}_{sys} = \vec{P}_{sysf} - \vec{P}_{sys1}$$

- System as a WHOLE will preserve momentum as long as not changed by a foreign object/force

$$\vec{P}_{sys} = \sum_{i=1}^{n} m_i \vec{V}_i = \sum_{i=1}^{n} m_i \frac{d\vec{r}_i}{dt} = \sum_{i=1}^{n} \frac{d\, m_i \vec{r}_i}{dt} = \frac{d}{dt} \sum_{i=1}^{n} m_i \vec{r}_i$$

$$\Rightarrow \sum_{i=1}^{n} m_i \frac{d}{dt} \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \sum_{i=1}^{n} m_i \cdot \frac{d\vec{r}_{cm}}{dt}$$

$$\vec{P}_{sys} = M_{sys} \frac{d\vec{r}_{cm}}{dt}$$

$$\sum F_{sys} = \frac{d\vec{P}_{sys}}{dt} = \frac{d}{dt} M_{sys} \frac{d\vec{r}_{cm}}{dt}$$

$$0 = \frac{d}{dt}\left(\frac{d\vec{r}_{cm}}{dt}\right)$$

Assuming Fsys is 0

Velocity of center of mass remains constant

Made with Goodnotes

Interactions:

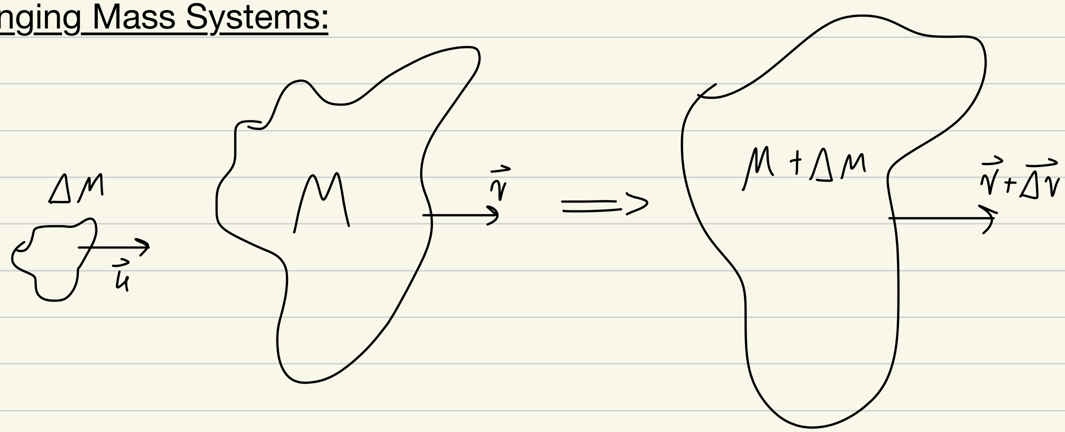
Explosion/Recoil:
- Shooting gun
- Football
- Moving with the same initial velocity —> interaction happens —> components of system separate

Perfectly Inelastic:
- Same final velocity

Elastic:
- Separate at start and separate at end
- Kinetic Energy is preserved

Inelastic:
- Same as elastic but energy is lost

<u>Changing Mass Systems:</u>



$$\Sigma P_1 = \Delta M \vec{u} + M\vec{v}$$

$$\Sigma P_f = (M + \Delta M)(\vec{v} + \Delta\vec{v}) = Mv + \Delta M \vec{v} + M\Delta\vec{v} + \Delta M \Delta\vec{v}$$

$$\Sigma \Delta\vec{p} = \cancel{Mv} + \Delta M\vec{v} + M\Delta\vec{v} + \Delta M\Delta\vec{v} - \Delta M\vec{u} - \cancel{M\vec{v}}$$

$$= \Delta M(\vec{v} - \vec{u}) + M\Delta\vec{v} + \Delta M\Delta\vec{v}$$

Assume $\Delta M <<< M$, $\Delta v <<< v$ $\therefore$ $\Delta M(\vec{v} - \vec{u}) + M\Delta\vec{v} + \cancel{\Delta M\Delta\vec{v}} = \Delta M(\vec{v} - \vec{u}) + M\Delta\vec{v}$

$$\Sigma F_{external} = \lim_{\Delta t \to 0} \frac{\Delta p}{\Delta t}$$

$$\Sigma F_{ext} = \frac{dM}{dt}(\vec{v} - \vec{u}) + M\frac{dv}{dt}$$

$$= \underbrace{-(\vec{u} - \vec{v})\frac{dM}{dt}}_{F_{thrust}} + M\frac{dv}{dt}$$

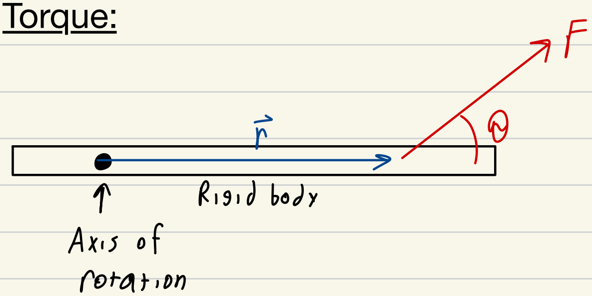
Look at this with the perspective of the bigger mass

$\vec{v}_{relative}$

<u>*NOTE IN ALL MOMENTUM EQUATIONS USE VELOCITY RELATIVE TO THIRD PARTY</u>

Rotational Kinematics

Torque:

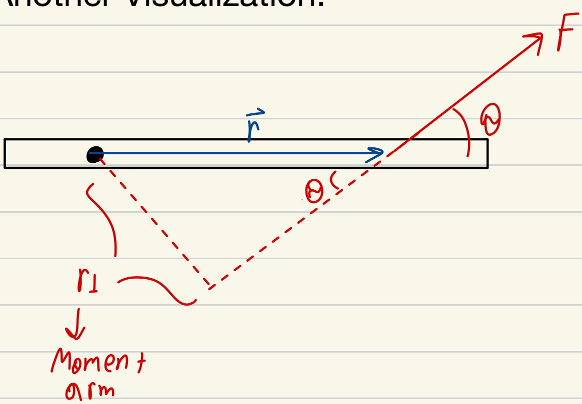


$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}|\,|\vec{F}| \sin\theta$$

$$= |\vec{r}|\,|\vec{F}_{\perp}|$$

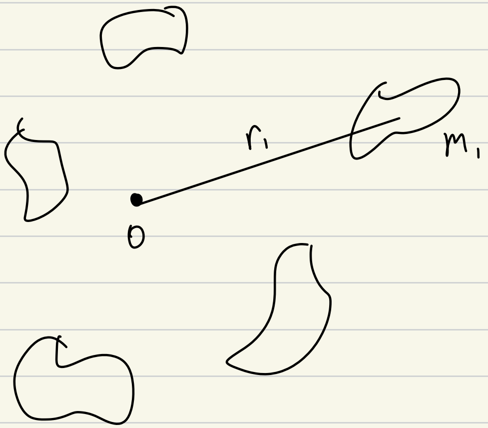
Another Visualization:



$$|\vec{\tau}| = |\vec{F}|\,|\vec{r}_{\perp}|$$

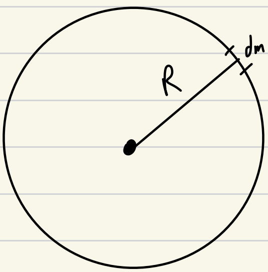
$$r_{\perp} = r \sin\theta$$

Moment of Inertia:



$$I_1 = m_1 r_1^2$$
$$I_2 = m_2 r_2^2$$

$$I_{sys} = \sum_{i=1}^{n} m_i r_i^2$$



$$dI = dm\, R^2$$

$$I = \int R^2\, dm$$

$$I = mR^2$$

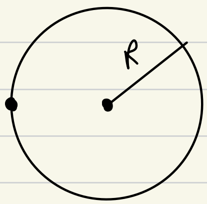
Parallel Axis Theorem:

$I_{cm} \rightarrow$ Rotational inertia about an axis passing through the center of mass

$I_p \rightarrow$ Rotational inertia about an axis parallel to the axis passing through the center of mass, and a distance d away from the cm axis

$$I_p = I_{cm} + Md^2$$

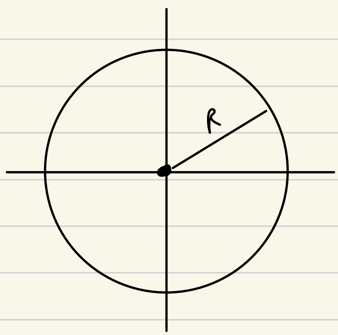
Ex.



$$I_P = MR^2 + MR^2$$
$$= 2MR^2$$

Perpendicular Axis Theorem (Only for 2-Dimensions):

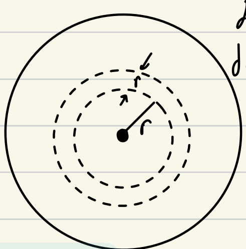
- For a 2D object, the moment of inertia about an axis passing through the cm perpendicular to the plane of the object is equal to the sum of the moments of inertia about 2 mutually perpendicular axes in the plane of the object is equal



$$I_{cm} = I_x + I_y$$

$$I_x = I_y$$
$$I_{cm} = 2I_y$$
$$MR^2 = 2I_y$$
$$I_y = \frac{1}{2}MR^2$$

Uniform Disc:

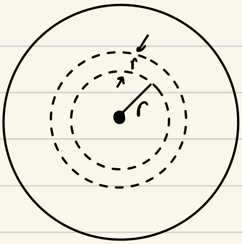


$$dA = \pi(r+dr)^2 - \pi r^2$$
$$dA = \pi(r^2 + 2rdr + \cancel{dr^2}) - \pi r^2$$
$$da = 2\pi r\,dr$$
or
$$A = \pi r^2$$
$$dA = 2\pi r\,dr$$

$$\frac{dm}{dA} = \frac{M}{\pi R^2}$$
$$dm = \frac{M}{\pi R^2}\,dA$$
$$dm = \frac{M}{\pi R^2}\cdot 2\pi r\,dr$$

$$\int dI = \int dm\,r^2$$
$$I = \int\left(\frac{M}{\pi R^2}\cdot 2\pi r\right) r^2\,dr$$
$$I = \frac{2\pi M}{\pi R^2}\int_0^R r\,dr\cdot r^2$$
$$I = \frac{1}{2}MR^2$$

Non-Uniform Disc:



σ = Density

$\sigma(r) = 3r^2$

$dm = \sigma\, dA = \sigma\, 2\pi r\, dr$

$dm = 3r^2 (2\pi r)\, dr$

$= 6\pi r^3\, dr$

$dI = dm\, r^2$

$\int dI = \int 6\pi r^5\, dr$

$I = [\pi r^6]_0^R$

$I = \pi R^6$

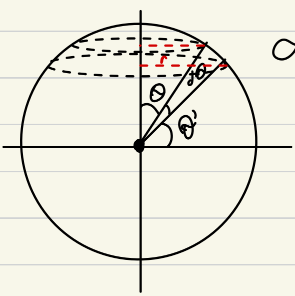
$M = \int 6\pi r^3\, dr$

$M = \frac{3\pi R^4}{2}$

$I = \pi R^6 \frac{M}{\frac{3\pi R^4}{2}}$

$I = \frac{2}{3} M R^2$

Spherical Shell:



$\sigma = \frac{M}{4\pi R^2} = \frac{M}{\text{Surf. Area}}$

$r = R\sin\theta$

$ds = R\, d\theta$

$dM = \sigma\, dA$

$= \sigma\, 2\pi r\, ds$

$= \sigma\, 2\pi R\sin\theta \cdot R\, d\theta$

$dI = \sigma\, 2\pi R\sin\theta \cdot R\, d\theta \cdot (R\sin\theta)^2$

$dI = \sigma\, 2\pi R^4 \sin^3\theta\, d\theta$

$I = 2\pi\sigma R^4 \int_0^\pi \sin^3\theta\, d\theta$

$= 2\pi\sigma R^4 \int_0^\pi \sin\theta(1-\cos^2\theta)\, d\theta$

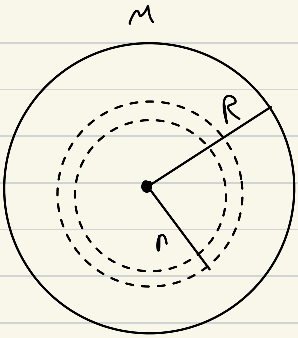
$= 2\pi\sigma R^4 \left(\int_0^\pi \sin\theta\, d\theta - \int_0^\pi \sin\theta\cos^2\theta\, d\theta \right)$

$= 2\pi\sigma R^4 \left([-\cos\theta]_0^\pi + \left[\frac{\cos^3\theta}{3}\right]_0^\pi \right)$

$= 2\pi\sigma R^4 \left(2 - \frac{2}{3}\right) = \frac{8\pi\sigma R^4}{3}$

$= \frac{2}{3} M R^2$

Solid Sphere:



$$I = \frac{2}{3} M R^2$$

Split into spherical shells

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\, dr$$

$$dm = \rho\, dV$$

$$dm = \rho\, 4\pi r^2\, dr$$

$$dI = \frac{2}{3}\, dm\, r^2$$

$$I = \int \frac{2}{3} r^2\, dm = \int \frac{2}{3} r^2 \rho\, 4\pi r^2\, dr$$

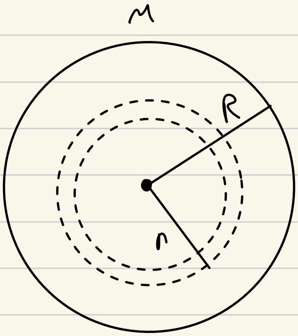
$$= \frac{8\pi\rho}{3} \int_0^R r^4\, dr$$

$$= \frac{8\pi\rho R^5}{15} = \frac{2}{5} M R^2$$

Verdict)

$I_{sphere} < I_{disc} < I_{shell} < I_{ring}$

Non-uniform Solid Sphere:



$$I = \frac{2}{3} M R^2$$

$$\rho = C e^{-kr}$$

$$dV = 4\pi r^2 dr$$

$$dm = \rho \, dV$$

$$dm = C e^{-kr} \cdot 4\pi r^2 dr$$

$$dI = \frac{2}{3} dm \, r^2$$

$$= \frac{2}{3} C e^{-kr} 4\pi r^2 dr \, r^2$$

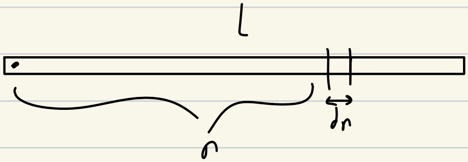
$$I = \frac{8\pi C}{3} \int_0^R e^{-kr} r^4 dr$$

r^4	$+$	e^{-kr}
$4r^3$	$-$	$-k e^{-kr}$
$12r^2$	$+$	$k^2 e^{-kr}$
$24r$	$-$	$-k^3 e^{-kr}$
24	$+$	$k^4 e^{-kr}$
0		$-k^5 e^{-kr}$

$$I = -kr^4 e^{-kr} - 4r^3 k^2 e^{-kr} - 12r^2 k^3 e^{-kr} - 24rk^4 e^{-kr} - 24k^5 e^{-kr}$$

Rod:

About a Pivot



$$\frac{M}{L} = \frac{dm}{dr}$$

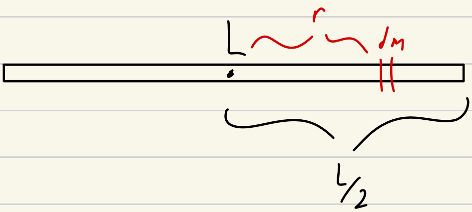
$$dm = \frac{M}{L}\,dr$$

$$dI = dm\,r^2$$

$$\int dI = \int_0^L \frac{M}{L} r^2\,dr$$

$$I = \frac{ML^3}{3L} = \frac{1}{3}ML^2$$

About Center of Mass:



$$\lambda = \frac{M}{L}$$

$$dI = dm\,r^2$$

$$dm = \frac{M}{L}\,dr \rightarrow dI = \frac{M}{L}\,dr\,r^2$$

$$I = \int_0^{L/2} \frac{M}{L} r^2\,dr$$

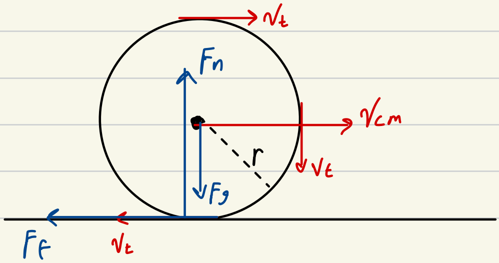
$$I = \frac{M}{L}\,\frac{L^3}{24}$$

$$I = \frac{1}{24}ML^2$$

$$\Sigma I = 2I$$

$$\Sigma I = \frac{1}{12}ML^2$$

Rotation:



$\sum \tau = I\alpha$

When $v_{cm} = v_t$:

Pure Rolling

$r\omega = v_{cm} = v_t$

ω = angular speed

$\sum F_x = -F_{fk} = -\mu_k m g$

$\sum F_y = F_n - F_g$

$F_n = mg$

$\cancel{m}a = -\mu_k \cancel{m} g$

$a = -\mu_k g$

$v_f = v_0 + at$

$v_f = v_0 - \mu_k g t$

$\vec{\tau} = \vec{r} \times \vec{F}$

$z_g = 0(: r = 0)$

$\tau_N = 0$

$\sum \tau = \tau_{Ffk} = \mu_k m g r = I\alpha$

$\alpha = \frac{m g \mu_k r}{I}$

Angular accel

$\omega_f = \cancel{\omega_i} + \alpha t$

$\omega_f = \frac{\mu_k m g r}{I} t$

$v_f = r\omega = v_0 - \mu_k g t$

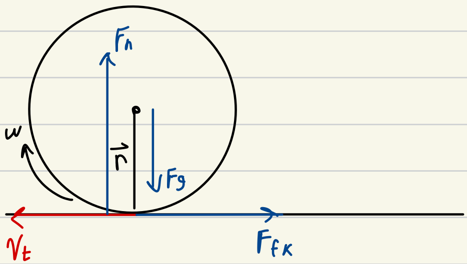
$r\left(\frac{\mu_k m g r}{I} t\right) = v_0 - \mu_k g t$

True at the instant of pure rolling

$t = \frac{v_0}{\mu_k g\left(\frac{mr^2}{I} + 1\right)}$

1) The ball initially only has Vcm
2) Vt is created from the torque of friction and then static friction causes pure rolling

Initially, just vcm, Ffk and no rolling
Second, Ffk SLOWS vcm and torque increases Vt
Once they meet eachother, it reaches pure rolling
After long period of time, vcm hits zero and its only Vt



This time, we start with tangential velocity

Giving the ball an initial spin

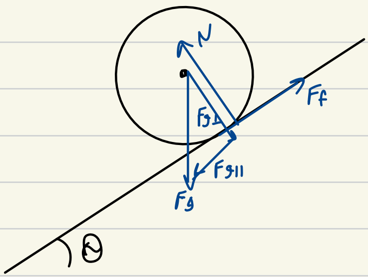
$$t = 0, \quad w = w_0$$
$$t = 0, \quad V_{cm} = 0$$

$$\sum \tau = -mg\mu_k r \qquad \sum F_{\parallel} = ma$$
$$I\alpha = -mg\mu_k r \qquad \cancel{m}g\mu_k = \cancel{m}a$$
$$\alpha = \frac{-mg\mu_k r}{I} \qquad a = g\mu_k$$

$$w_f = w_i + \alpha t \qquad V_f = g\mu_k t$$
$$w_f = w_0 - \frac{mg\mu_k r}{I} t$$

$$V_f = r w$$
$$g\mu_k t = \left(w_0 - \frac{mg\mu_k r}{I} t\right) r$$

$$t = \frac{w_0 r}{\mu_k g\left(\frac{mR^2}{I} + 1\right)}$$



$v_{cm} > v_t$

$\therefore F_f \nearrow$

Friction will point opposite the NET velocity

$t = 0,\ v_{cm} = 0,\ v_t = 0$

$\Sigma \tau = \tau_{F_{fk}}$

$I\alpha = mg\cos\theta\,\mu_s r$

$\alpha = \dfrac{mg\cos\theta\,\mu_s r}{I}$

$\omega_f = \dfrac{mg\cos\theta\,\mu_s r}{I} t$

$\Sigma F = ma$

$\cancel{m}g\sin\theta - \cancel{m}g\cos\theta\,\mu_s = \cancel{m}a$

$a = g\sin\theta - g\cos\theta\,\mu_s$

$v_f = (g\sin\theta - g\cos\theta\,\mu_s)t$

$v_f = r\omega$

$(\cancel{g}\sin\theta - \cancel{g}\cos\theta\,\mu_k)\cancel{t} = \dfrac{m\cancel{g}\cos\theta\,\mu_s r^2}{I}\cancel{t}$

$\sin\theta - \cos\theta\,\mu_k = \dfrac{m\cos\theta\,\mu_k r^2}{I}$

$\sin\theta - \mu_k\cos\theta = \dfrac{{}_s m\cos\theta\, r^2}{I}$

$\tan\theta - \mu_k = \dfrac{mr^2\mu_s}{I}$

$\tan\theta = \dfrac{mr^2\mu_s}{I} + \mu_s$

$\boxed{\theta = \tan^{-1}\left(\dfrac{mr^2\mu_s}{I} + \mu_s\right)} \implies \mu_s \geq \dfrac{I}{mr^2 + I}\tan\theta$

If we dont sub for Ff and solve for it:

$F_f = \dfrac{g\sin\theta}{\frac{r^2}{I} + \frac{1}{m}}$

If $F_f > \mu_s F_n$

Unable to transition to pure rolling

Using Energy:

$$KE = KE_{tr} + KE_{rot}$$

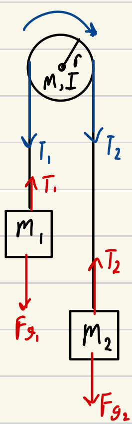
$$U_{gi} + KE_{i\,tr} + KE_{i\,rot} = U_{gf} + KE_{f\,tr} + KE_{f\,rot}$$

$$mg(h+r) + 0 + 0 = mgr + \frac{1}{2} m v_f^2 + \frac{1}{2} I \omega_f^2$$

$$2mgh = \left(m + \frac{I}{r^2}\right) v_f^2$$

$$v_f^2 = \frac{2mgh}{m + \frac{I}{r^2}}$$

Pulley With Mass:



$$m_2 > m_1$$

$$* \; v_t = \omega R$$

$$\frac{dv}{dt} = R\frac{d\omega}{dt}$$

$$a = R\alpha$$

No slipping

$$\Sigma \tau = T_2 R - T_1 R = I\alpha = I\left(\frac{a}{R}\right) \Rightarrow (T_2 - T_1)R = I\frac{a}{R}$$

$$(T_2 - T_1) = I\frac{a}{R^2}$$

$$T_1 - m_1 g = m_1 a$$

$$m_2 g - T_2 = m_2 a$$

$$m_2 g - m_1 g + T_1 - T_2 = m_{sys} a$$

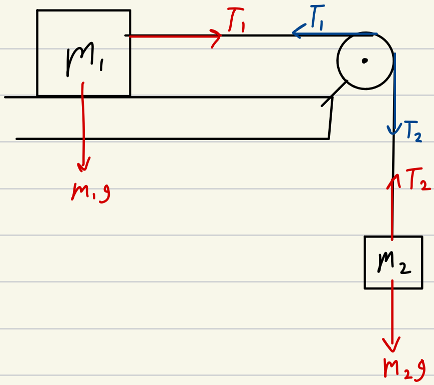
Two diff tensions

$$T_1 - T_2 + (m_2 - m_1)g = (m_1 + m_2)a$$

$$T_2 - T_1 = (m_2 - m_1)g - (m_1 + m_2)a$$

$$I\frac{a}{R^2} = (m_2 - m_1)g - (m_1 + m_2)a$$

$$\boxed{a = \frac{(m_2 - m_1)g}{m_1 + m_2 + I/R^2}}$$



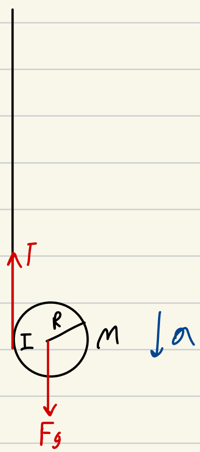
$$\sum \tau = T_2 R - T_1 R = I\alpha = I\frac{a}{R}$$
$$R(T_2 - T_1) = I\frac{a}{R}$$
$$(T_2 - T_1) = I\frac{a}{R^2}$$

$$m_2 g - T_2 + T_1 = (m_1 + m_2)a$$

$$m_2 g - (T_2 - T_1) = (m_1 + m_2)a$$
$$m_2 g - I\frac{a}{R^2} = (m_1 + m_2)a$$
$$m_2 g = (m_1 + m_2)a + I\frac{a}{R^2}$$
$$m_2 g = a\left((m_1 + m_2) + \frac{I}{R^2}\right)$$

$$\boxed{a = \frac{m_2 g}{(m_1 + m_2) + I/R^2}}$$

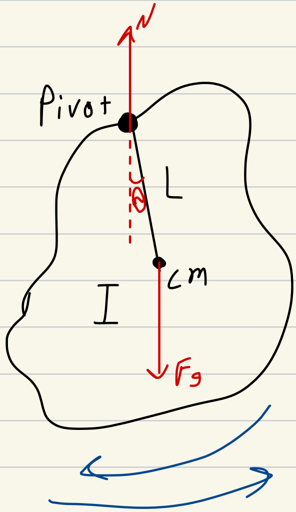
Yo-Yo:



$$\sum F = Mg - T = Ma$$
$$T = M(g - a) \longrightarrow I\frac{a}{R^2} = M(g - a)$$
$$I\frac{a}{R^2} + Ma = Mg$$
$$a\left(\frac{I}{R^2} + M\right) = Mg$$
$$\boxed{a = \frac{Mg}{\frac{I}{R^2} + M}}$$

$$\sum \tau = T \cdot R = I\alpha$$
$$TR = I\frac{a}{R}$$
$$T = I\frac{a}{R^2}$$

Physical Pendulum:



$$\sum \tau = F_g \sin\theta \, \ell = I\alpha$$

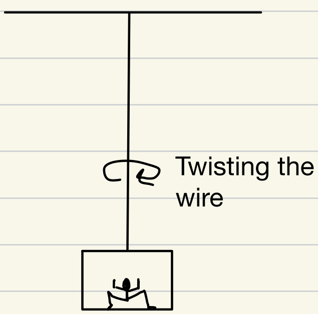
$$I \frac{d^2\theta}{dt^2} = -mgL\sin\theta$$

$$\frac{d^2\theta}{dt^2} = \frac{-mgL\sin\theta}{I}$$

$\theta \leq 10^\circ$

SHM

Torsional Pendulum:



$$\tau = -k\theta$$

$\downarrow$

Restoring Force

$$I\alpha = -k\theta$$

$$\frac{d^2\theta}{dt^2} = \frac{-k\theta}{I}$$

SHM

Angular Momentum

$$\sum \tau_{External} = \frac{d\vec{L}}{dt}$$

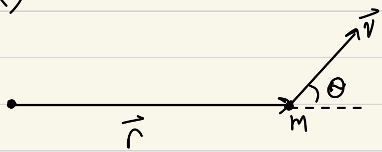
$\vec{L}$ = Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

If rotation is happening on a symmetry axis for a rigidbody:

$$L = I\vec{\omega}$$

Ex)



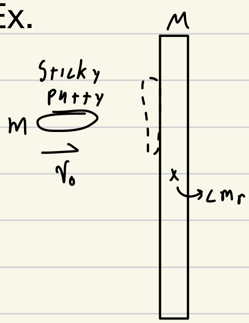
$$p = m\vec{v}$$

$$\vec{L} = \vec{r} \times m\vec{v}$$

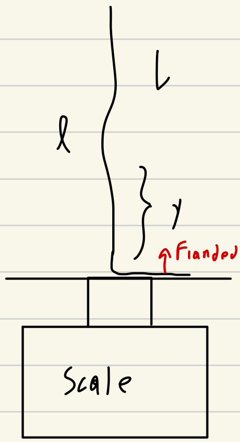
$$|\vec{L}| = mrv\sin\theta$$

- Both linear and angular momentum conserved
- Angular Momentum conserved about the CENTER OF MASS
- If its a pivoted object then momentum is conserved about the pivot

Ex.



Chain on A Scale Ex.



$\lambda = \frac{M}{L}$

L, M

$v_1 = \sqrt{2gy}$

$p_1 = dm\sqrt{2gy}$

$= \lambda\, dy \sqrt{2gy}$

$p_f = 0$

$dp = -\lambda\, dy\sqrt{2gy} = \Delta p$

Impulse from scale

$dp = \lambda\, dy \sqrt{2gy}$

$$\frac{dp}{dt} = \lambda \frac{dy}{dt}\sqrt{2gy} = \lambda\sqrt{2gy}\sqrt{2gy}$$

$F = \lambda 2gy$

Force required to stop the chain

$$\left\{ F_{scale} = F_{landed} + F \right.$$

$= \lambda y g + 2\lambda g y$

$= 3\lambda g y$

$= 3\frac{M}{L} g y$

Scale reading when y length falls